Part A

I. Answer all the multiple-choice questions: **1 x 15 = 15**

1. A relation R in a set A is called Reflexive relation if

a) $(a,a) \in R$ for all $a \in A$
b) $(a,a) \in R$ for atleast one $a \in A$
c) $(a,a) \in R$ implies $(b,a) \in R$
d) $(a,a) \in R$ and $(b,c) \in R$ implies $(a,c) \in R$

Ans: a)

2. The principal value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$ is

a) $\frac{\pi}{2}$ b) $\frac{\pi}{3}$ c) $\frac{\pi}{4}$ d) $\frac{\pi}{6}$

Ans: c)

3. Match List – I with List – II

List – I	List - II
A) Domain of $\sin^{-1}$ x	i) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$
B) Range of $\tan^{-1}$ x	ii) [0, π]
C) Range of $\cos^{-1}$ x	iii) [-1, 1]

Choose the correct answer from the options given below.

a) A -i, B – ii, C – iii
b) A -iii, B – ii, C – i
c) A -ii, B – i, C – iii
d) A -iii, B – i, C – ii

Ans: d)

4. For a 2 x 2 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = 2i - j$ then A is equal to

a) $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ d) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Ans: b)

5. Let A be a non-singular matrix of order 3 x 3, then $|\text{adj A}|$ is equal to

a) $|A|$ b) $3|A|$ c) $|A|^3$ d) $|A|^2$

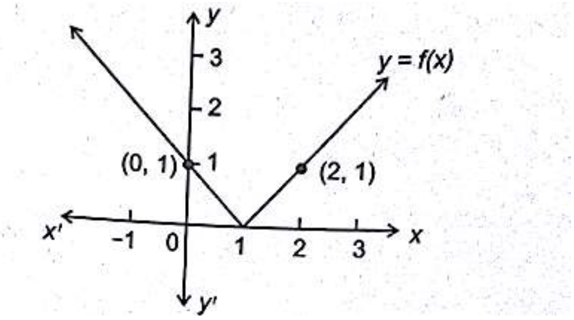
Ans: d)

6. If f (x) = cos 2x, then $f'\left(\frac{\pi}{4}\right)$ is

a) 2 b) -2 c) $\sqrt{2}$ d) $-\sqrt{2}$

Ans: b)

7. For the given figure consider the following statements 1 and 2



Statement 1: Left hand derivative of y = f(x) at x = 1 is -1.
Statement 2: The function y = f (x) is differentiable at x = 1.
Then which of the following are true?
a) Statement 1 is true, statement 2 is false
b) Statement 1 is false, statement 2 is true
c) Both statements 1and 2 are true
d) Both statements 1and 2 are false
Ans: a)

8. The absolute maximum value of the function f given by $f(x) = x^3$, $x \in [-2, 2]$ is
a) 2 b) 0 c) −2 d) 8
Ans: d)

9. $\int e^x (\sin x - \cos x)dx$ is
a) $-e^x \cos x$ b) $e^x \cos x$ c) $e^x \sin x$ d) $e^x \sin^2 x$
Ans: a)

10. The degree of differential equation $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + e^{\frac{dy}{dx}} = 0$ is
a) 1 b) 3 c) 2 d) not defined
Ans: d)

11. The direction cosines of the vector $\vec{a} = \hat{i} - j + 2k$ are
a) $\frac{1}{\sqrt{5}}, \frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}$ b) $\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$ c) $\frac{1}{6}, \frac{-1}{6}, \frac{2}{6}$ d) $\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$
Ans: b)

12. The angle between two vectors $\vec{a}$ and $\vec{b}$ with $|\vec{a}| = \sqrt{3}$, $|\vec{b}| = 2$ and $\vec{a} \cdot \vec{b} = \sqrt{6}$ is
a) $\frac{\pi}{6}$ b) $\frac{\pi}{3}$ c) $\frac{\pi}{4}$ d) $\frac{\pi}{2}$
Ans: c)

13. The equation of y-axis in space is
a) x = 0, y = 0 b) x = 0, z = 0 c) y = 0, z = 0 d) y = 0
Ans: b)

14. If $P(A) = \frac{1}{2}$, $P(B/A) = \frac{2}{3}$ then $P(A \cap B)$ is
a) $\frac{1}{3}$ b) $\frac{1}{2}$ c) 1 d) $\frac{3}{5}$
Ans: a)

15. Assertion [A]: For two events E and F if $P(E)=\frac{1}{5}$, $P(F)=\frac{1}{2}$ and $P(E|F)=\frac{1}{5}$ then E and F are independent events.

Reason [R]: If E and F are two independent events then P(F|E) = P(F).

Then which of the following are true?

a) [A] is true but [R] is false
b) Both [A] and [R] are false
c) Both [A] and [R] are true
d) [A] is false but [R] is true

Ans: c)

II. Fill in the blanks by choosing appropriate answer from those given in the brackets:

[0, 2,1, $\frac{5}{9}$, -1, 6] **1 x 5 = 5**

16. The value of $\cos\left(\sec^{-1}(2)-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$ is ____

Ans: 1

17. If $y=\sin^{-1}(\cos x)$ then $\frac{dy}{dx}=$ ____

Ans: -1

18. The value of $\int_{7}^{13} 1dx=$ ___

Ans: 6

19. The projection of vector $\hat{i}+j$ along the vector $\hat{i}-j$ is____

Ans: 0

20. If $P(A\cap B)=\frac{4}{13}$ and $P(B)=\frac{9}{13}$ then $P(A'|B)=$

Ans: $\frac{5}{9}$

Part B

III. Answer any SIX of the following questions: 2 x 6 = 12

21. Find the equation of the line joining (1, 2) and (3, 6) using determinants.

Ans: We know that Area of triangle

$$=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

If area of triangle = 0.

Then it forms a straight line.

$$\therefore (x_1 y_1)=(1,2);(x_2 y_2)=(3,6);(x_3 y_3)=(x,y)$$

$$\therefore \quad \Delta = \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ x & y & 1 \end{vmatrix}$$

$$0 = \frac{1}{2}\{1(6-y) - 2(3-x) + 1(3y-6x)\}$$

$$0 = \frac{1}{2}[6 - y - 6 + 2x + 3y - 6x]$$

$$0 = \frac{1}{2}[-4x + 2y]$$

$$0 = \frac{1}{2}[2(-2x + y)]$$

$$0 = -2x + y$$

$$2x - y = 0 \quad \text{or} \quad 2x = y$$

22. If $\sqrt{x} + \sqrt{y} = \sqrt{10}$, Show that $\frac{dy}{dx} + \sqrt{\frac{y}{x}} = 0$

Ans:

$\sqrt{x} + \sqrt{y} = \sqrt{10}$

Differentiate both sides w.r.t to x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{1}{2}\left[\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}\frac{dy}{dx}\right] = 0$$

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{1}{\sqrt{y}}\frac{dy}{dx} = -\frac{1}{\sqrt{x}}$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

$$\therefore \quad \frac{dy}{dx} + \sqrt{\frac{y}{x}} = 0$$

23. A balloon which is always remains spherical has a variable radius. Find the rate at which its volume is increasing with radius when the radius is 10 cms.

Ans: $V = \frac{4}{3}\pi r^3$

$$\frac{dv}{dr} = \frac{4}{3}\pi\left(3r^2\right)$$

$$\left.\frac{dv}{dr}\right|_{r=10} = \pi\left(4(100)\right) = 400\pi \ cm^3 / cm$$

24. Find the intervals in which the function f is given by $f(x) = 4x^3 - 6x^2 - 72x + 30$ is decreasing.

Ans:

We have $f(x) = 4x^3 - 6x^2 - 72x + 30$

$f^1(x) = 12x^2 - 12x - 72$

$=12\left(x^2-x-6\right)=12(x-3)(x+2)$

Now $f^1(x)=0 \Rightarrow 12(x-3)(x+2)=0$

$\Rightarrow x=3,\ x=-2$

divides the real line in to three disjoint interval $(-\infty,-2),(-2,3),(3,\infty)$

Interval	sign of $f^1(x)$	Nature of function f
$(-\infty,-2)$	$(-)(-)>0$	f is strictly increasing
$(-2,3)$	$(-)(+)<0$	f is strictly decreasing
$(3,\infty)$	$(+)(+)>0$	f is strictly increasing

Thus **f** is strictly decreasing in (- 2, 3).

25. Find $\int \mathbf{\log(\sin x)\cot x.dx}$

Ans: $I=\int \log(\sin x)\cot x.dx$

Put $\log\sin x = t$

D.w.r.to x

$\frac{\cos x}{\sin x}=\frac{dt}{dx}$

$\cot x.dx = dt$

$=\int t.dt$

$=\frac{t^2}{2}+C$

$=\frac{(\log(\sin x))^2}{2}+C$

26. Verify that the function $y=a\sin x+b\cos x$ **is a solution of differential equation** $\frac{d^2y}{dx^2}+y=0$

Ans: $y=a\cos x+b\sin x$

$\frac{dy}{dx}=-a\sin x+b\cos x$

$\frac{d^2y}{dx^2}=-(a\cos x+b\sin x)$

$\frac{d^2y}{dx^2}=-y$

$\frac{d^2y}{dx^2}+y=0$

27. If $\vec{a}=\hat{i}+j+k,\vec{b}=2\hat{i}-j+3k$ *and* $\vec{c}=\hat{i}-2j+k$ **then find unit vector parallel to the vector** $2\vec{a}-\vec{b}+3\vec{c}$**.**

Ans: $\vec{a}=\hat{i}+j+k,\vec{b}=2\hat{i}-j+3k,\vec{c}=\hat{i}-2j+k$

$\vec{r}=2\vec{a}-\vec{b}+3\vec{c}=3\hat{i}-3j+2k$

$\hat{r}=\frac{3\hat{i}-3j-2k}{\sqrt{22}}$

28. If the lines $\frac{\mathbf{x-1}}{\mathbf{-3}}=\frac{\mathbf{y-2}}{\mathbf{2k}}=\frac{\mathbf{z-3}}{\mathbf{2}}$ **&** $\frac{\mathbf{x-1}}{\mathbf{3k}}=\frac{\mathbf{y-1}}{\mathbf{1}}=\frac{\mathbf{z-6}}{\mathbf{-5}}$ **are perpendicular find k.**

Ans: Given equation of the line is of the form $\frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ & $\frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$ Two lines are perpendicular if

$a_1a_2+b_1b_2+c_1c_2=0$

$$(-3)(3k)+(2k)(1)+2(-5)=0$$

$$-9k+2k-10=0 \Rightarrow \frac{-10}{7}=k$$

29. An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after other without replacement. What is the probability that Both drawn balls are black?

Ans:

Let E and F denote respectively the events that first and second ball drawn are black. We have to find $P(E\cap F)$

Now, $P(E) = P(\text{black ball in first draw}) = \frac{10}{15}$

Therefore, the probability that the second ball drawn is black, given that the ball in the first drawn is black, is nothing but the conditional probability of F given that E has occurred

i.e, $P(F/E) = \frac{9}{14}$

By multiplication rule of probability, we have

$$P(E\cap F) = P(E)P(F|E)$$

$$= \frac{10}{15}\times\frac{9}{14} = \frac{3}{7}$$

Part C

IV. Answer any SIX of the following questions: 3 x 6 = 18

30. Check whether the relation R in R defined by $R=\{(a,b);a\le b^3\}$ is reflexive , symmetric and transitive.

Solution: (i) We have $R=\{(a,b);a\le b^3\}$

Consider $\left(\frac{1}{2},\frac{1}{2}\right)$

clearly $\frac{1}{2}\le\left(\frac{1}{2}\right)^3$ is not true

Thus $\left(\frac{1}{2},\frac{1}{2}\right)\notin R$ that is R is not reflexive.

(ii) Clearly $(1,2)\in R$ but $(2,1)\notin R$

i.e, $1\le 2^3$ but $2\le 1^3$ is not true.

R is not symmetric.

(iii) $(10,3)\in R$ $(3,2)\in R$ but $(10,2)\notin R$

Now, we have $10\le(3)^3$ thus $(10,3)\in R$

And $3\le 2^3$ thus $(3,2)\in R$ But $(10,2)\notin R$

$\because$ $10\le 2^3$ is not true

Thus R is not transitive.

31. Prove that $\tan^{-1}\left(\frac{63}{16}\right)=\sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{3}{5}\right)$

Solution

$$\text{R H S } =\sin^{-1}\left(\frac{5}{13}\right)+\cos^{-1}\left(\frac{3}{5}\right)$$

$$=\tan^{-1}\left(\frac{5}{12}\right)+\tan^{-1}\left(\frac{4}{3}\right)$$

$$=\tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}.\frac{4}{3}}\right)$$

$$=\tan^{-1}\left[\frac{\frac{15+48}{\cancel{36}}}{\frac{36-20}{\cancel{36}}}\right]=\tan^{-1}\left[\frac{63}{16}\right]$$

32. Express $A=\begin{bmatrix}1 & 5\\ -1 & 2\end{bmatrix}$ as sum of symmetric and skew symmetric matrix.

Solution:

Let $A=\begin{bmatrix}1 & 5\\ -1 & 2\end{bmatrix}$, $\quad A'=\begin{bmatrix}1 & -1\\ 5 & 2\end{bmatrix}$

$P=\frac{1}{2}(A+A')=\frac{1}{2}\begin{bmatrix}2 & 4\\ 4 & 4\end{bmatrix}=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}$

$P'=\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}=P$ $\quad\therefore\quad P=\frac{1}{2}(A+A')$ is a symmetric matrix

$\therefore\quad A-A'=\begin{bmatrix}3 & 5\\ 1 & -1\end{bmatrix}-\begin{bmatrix}3 & 1\\ 5 & -1\end{bmatrix}=\begin{bmatrix}0 & 4\\ -4 & 0\end{bmatrix}$

$Q=\frac{1}{2}(A-A')=\frac{1}{2}\begin{bmatrix}0 & 4\\ -4 & 0\end{bmatrix}=\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}$

Now $\quad Q'=\begin{bmatrix}0 & -2\\ 2 & 0\end{bmatrix}=-Q$

$Q=\frac{1}{2}(A-A')$ is a skew symmetric matrix

Now $\quad P+Q=\begin{bmatrix}3 & 3\\ 3 & -1\end{bmatrix}+\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}=A$

33. Find $\frac{dy}{dx}$, if $x=a\left[\cos t+\log\tan\left(\frac{t}{2}\right)\right]$ and $y=a\sin t$.

Solution: $x=a\left[\cos t+\log\tan\left(\frac{t}{2}\right)\right]$

Differentiate w. r.to t

$$\frac{dx}{dt}=a\left\{-\sin t+\frac{1}{\tan t/2}\times\sec^2\frac{t}{2}\times\frac{1}{2}\right\}$$

$$= a\left\{-\sin t + \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} \times \frac{1}{\cos^2\frac{t}{2}} \times \frac{1}{2}\right\}$$

$$= a\left\{-\sin t + \frac{1}{\sin t}\right\} = a\left\{\frac{-\sin^2 t + 1}{\sin t}\right\} = a\,\frac{\cos^2 t}{\sin t}$$

$y = a \sin t$

Differentiate w.r.to t

$$\frac{dy}{dt} = a\cos t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{a\frac{\cos^2 t}{\sin t}} = \frac{\sin t}{\cos t} = \tan t$$

34. Find two positive numbers x & y such that $x + y = 60$ and xy^3 is maximum.

Solution : Let $P = xy^3$ $\qquad x + y = 60$ (given)

$= (60 - y)y^3 \qquad (\because y = 60 - x)$

$P = 60y^3 - y^4$

Differentiate w. r. to y

$$\frac{dP}{dy} = 60 \times 3y^2 - 4y^3$$

$$= 180y^2 - 4y^3$$

$$\frac{d^2P}{dy^2} = 360y - 12y^2 = y(360 - 12y)$$

For the value to be max/ min $\frac{dP}{dy} = 0$

$$\frac{dP}{dy} = 0 \quad \Rightarrow \quad 180y^2 - 4y^3 = 0 \quad \Rightarrow \quad 180y^2 = 4y^3$$

$180 = 4y$

$y = \frac{180}{4} = 45$

$x = 60 - y = 60 - 45 = 15$

$\therefore$ P is maximum

when $x = 45, y = 15$ or $x = 15, y = 45.$

35. Evaluate $\int \frac{2x}{x^2 + 3x + 2}.dx$

Solution: $\int \frac{2x}{x^2 + 3x + 2}.dx$

$$\int \frac{2x}{(x+2)(x+1)}.dx$$

$$\frac{2x}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$$

$\Rightarrow \quad 2x = A(x + 1) + B(x + 2)$

Put $x = -1$ we get $B = -2$

Put $x = -2$ we get $A = 4$

$$\therefore \int \frac{2x}{(x+2)(x+1)}dx$$

$$= \int\left(\frac{4}{x+2} - \frac{2}{x+1}\right)dx$$

$= 4 \log |x + 2| - 2 \log |x + 1| + C$

36. Find the area of the triangle ABC where position vectors of A, B and C are $\hat{i} - \hat{j} + 2\hat{k},\ 2\hat{j} + \hat{k}$ and $\hat{j} + 3\hat{k}$ respectively.

Solution : Given

$\overrightarrow{OA} = \hat{i} - \hat{j} + 2\hat{k},\ \overrightarrow{OB} = 2\hat{j} + \hat{k},\ \overrightarrow{OC} = \hat{j} + 3\hat{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -\hat{i} + 3\hat{j} - \hat{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = -\hat{i} + 2\hat{j} + \hat{k}$

Area of the triangle is $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

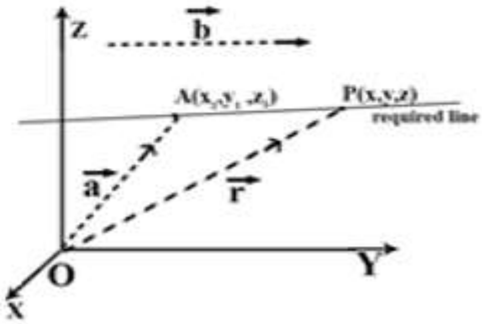
Now, $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 3 & -1 \\ -1 & 2 & 1 \end{vmatrix} = \hat{i}(3+2) - \hat{j}(-1-1) + \hat{k}(-2+3) = 5\hat{i} + 2\hat{j} + \hat{k}$

$\therefore \quad \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{25+4+1} = \sqrt{30}$

Thus the required area is $\frac{1}{2}\sqrt{30}$.

37. Derive the equation of line in space passing through a point and parallel to the vector both in vector and Cartesian form.

Ans:



Let $\vec{a}$ be the position vector of the given point A with respect to the origin O. let 'l' be the line passes through the point A and is parallel to a given vector $\vec{b}$. Let $\vec{r}$ be the position vectors of any point P on the line.

Then $\overrightarrow{AP}$ is parallel to $\vec{b}$,

We have $\overrightarrow{AP} = \lambda\vec{b}$

$\Rightarrow \quad \overrightarrow{OP} - \overrightarrow{OA} = \lambda\vec{b}$

$\Rightarrow \quad \vec{r} - \vec{a} = \lambda\vec{b}$

$\Rightarrow \quad \vec{r} = \vec{a} + \lambda\vec{b}$

This gives the position vector of any point P on the line.

Hence it is called vector equation of the line.

38. Box-I contains 2 gold coins, while another Box-II contains 1 gold coin and 1 silver coin. A person chooses a box at random and takes out a coin. If the coin is of gold, what is the probability that the other coin in the box is also of gold?

Solution : Let E_1 be the event of choosing Box-I

Let E_2 be the event of choosing Box-II.

Then $P(E_1) = P(E_2) = \frac{1}{2}$

Also, A is the event that the coin drawn is of gold.

Then, $P(A|E_1) = P$ (A gold coin from Box-I)

$= \frac{2}{2} = 1$

$P(A|E_2) = P$ (A gold coin from Box-II) $= \frac{1}{2}$

Now, the probability that the other coin in the box is of gold

The probability that gold coin is drawn from the

Box-I $= P(E_1|A)$

By using Baye's Theorem,

$$P(E_1|A) = \frac{P(E_1)P(A|E_1)}{P(E_1)P(A|E_1) + P(E_2)P(A|E_2)}$$

$$P(E_1|A) = \frac{\frac{1}{2}\times 1}{\frac{1}{2}\times 1 + \frac{1}{2}\times\frac{1}{2}}$$

$$\Rightarrow \quad P(E_1|A) = \frac{2}{3}$$

Part D

V. Answer any FOUR of the following questions: 5 x 4 = 20

39. If $A = R - \{3\}$ and $B = R - \{1\}$ and $f : A \to B$ is a function defined by $f(x) = \left(\frac{x-2}{x-3}\right)$. Is f one - one and onto? Justify your answer.

Solution:

Given $f(x) = \left(\frac{x-2}{x-3}\right)$

let $x_1, x_2 \in A = R - \{3\}$

$f(x_1) = f(x_2)$

$\Rightarrow \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$

$\Rightarrow (x_1 - 2)(x_2 - 3) = (x_2 - 2)(x_1 - 3)$

$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 3x_2 - 2x_1 + 6$

$\Rightarrow \cancel{x_1x_2} - 3x_1 - 2x_2 + 6 = \cancel{x_1x_2} - 3x_2 - 2x_1 + 6$

$3x_2 - 2x_2 = 3x_1 - 2x_1$

$\Rightarrow x_2 = x_1 \Rightarrow x_1 = x_2$

$\therefore f$ is one-one

Let $y \in B = R - \{1\}$ and let $f(x) = y$

$$\Rightarrow \frac{x-2}{x-3} = y$$

$$\Rightarrow x-2 = xy-3y \Rightarrow x-xy = 2-3y$$

$$\Rightarrow (1-y) = 2-3y \Rightarrow x = \frac{2-3y}{1-y} \in A$$

$\therefore$ corresponding to each $y \in B$ there exists $\left(\frac{2-3y}{1-y}\right) \in A$ such that

$$f\left(\frac{2-3y}{1-y}\right) = \frac{\frac{2-3y}{1-y}-2}{\frac{2-3y}{1-y}-3} = \frac{2-3y-2+2y}{2-3y-3+3y} = \frac{-y}{-1} = y$$

$\therefore f$ is onto

Hence, f is one-one and onto.

Hence it is a bijective function.

40. For the matrices A and B, verify that $(AB)' = B'A'$ where $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$

Solution: $A.B = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}\begin{bmatrix} -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}$

$$\therefore \quad (AB)' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix} ---------(1)$$

$$B' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, A' = \begin{bmatrix} 1 & -4 & 3 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix} ----(2)$$

From (1) and (2) $(AB)' = B'A'$.

41. Solve the following system of linear equations by matrix method

$4x+3y+2z=60, 2x+4y+6z=90, 6x+2y+3z=70$.

Solution :

This system can be written as AX = B, where

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{vmatrix} = 4(12-12) - 3(6-36) + 2(4-24) = 90 - 40 = 50 \neq 0$$

$\therefore$ Hence, A is nonsingular and so its inverse exists.

To find co-factor

$A_{11} = 0,\quad A_{12} = 30\quad A_{13} = -20$

$A_{21} = -5,\quad A_{22} = 0\quad A_{23} = 10$

$A_{31} = 10\quad, A_{32} = -20\quad A_{33} = 10$

$$\text{Co-factor matrix } A = \begin{bmatrix} 0 & 30 & -20 \\ -5 & 0 & 10 \\ 10 & -20 & 10 \end{bmatrix}$$

$$\therefore \quad \text{Adj } A = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{|A|}(\text{Adj } A) = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

As , $AX = B \Rightarrow X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 6 \\ 9 \\ 7 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 0-45+70 \\ 180+0-140 \\ -120+90+70 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 25 \\ 40 \\ 40 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

x = 5, y = 8 and z = 8

42. If $y = \left(\tan^{-1}x\right)^2$ then show that $\left(x^2+1\right)^2\frac{d^2y}{dx^2} + 2x\left(x^2+1\right)\frac{dy}{dx} = 2.$

Soln: $y = \left(\tan^{-1}x\right)^2$

Differentiate. w.r.t. x

$$\frac{dy}{dx} = 2\left(\tan^{-1}x\right)\times\frac{1}{1+x^2}$$

By cross multplying

$$\left(1+x^2\right)\frac{dy}{dx} = 2\left(\tan^{-1}x\right)$$

Again Diff. w. r. t. x on both sides

$$\left(1+x^2\right)\frac{d^2y}{dx^2} + \frac{dy}{dx}.2x = \frac{2}{1+x^2}$$

multiply $\left(1+x^2\right)$ on bothsides

$$\left(1+x^2\right)^2\frac{d^2y}{dx^2} + 2x\left(1+x^2\right)\frac{dy}{dx} = 2$$

43. Find the integral value of $\int \frac{dx}{a^2+x^2}$ and hence evaluate $\int \frac{1}{x^2-6x+13}dx$

Solution:

Let $I = \int \frac{dx}{a^2+x^2}$

Put $x = a\tan\theta$

$\theta = \tan^{-1}\frac{x}{a}$

D.w.r to θ

$dx = a\sec^2\theta \, d\theta$

$= \int \frac{a\sec^2\theta \, d\theta}{a^2+a^2\tan^2\theta}$

$= \int \frac{a\sec^2\theta \, d\theta}{a^2(1+\tan^2\theta)}$

$= \frac{1}{a}\int d\theta$

$= \frac{1}{a}\theta + c$

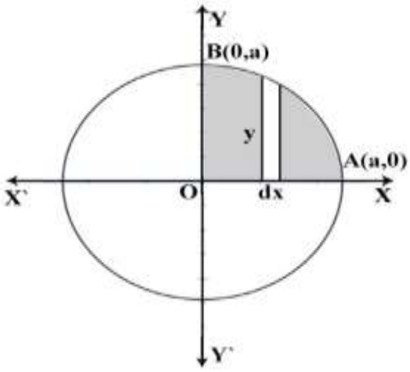
$= \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

We have $x^2-6x+13=(x-3)^2+2^2$

So $\int \frac{1}{x^2-6x+13}dx = \int \frac{1}{(x-3)^2+2^2}dx = \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+c$

44. Find the area of circle $x^2+y^2=a^2$ by method of integration

Soln:



Area of circle=4{area of the region of AOBA}

Area of AOBA = $\int_0^a y\, dx$

Now, $x^2+y^2 \Rightarrow y^2 = a^2 - x^2$

Area of AOBA = $\int_0^a \sqrt{a^2-x^2}\, dx$

$= \left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$

$= \left[\left(\frac{a}{2}\times 0+\frac{a^2}{2}\sin^{-1}1\right)-0\right] = \frac{\pi a^2}{4}$ squre units.

Area of circle $= 4\frac{\pi a^2}{4} = \pi a^2$ square units

45. Solve the differential equation $\cos^2 x.\frac{dy}{dx}+y=\tan x\left(0\le x<\frac{\pi}{2}\right)$

Soln: We have $\cos^2 x.\frac{dy}{dx}+y=\tan x\left(0\le x<\frac{\pi}{2}\right)$

divided by $\cos^2 x$ we get

$$\frac{dy}{dx} + y.\sec^2 x = \tan x.\sec^2 x$$

compare with $\frac{dy}{dx} + py = Q$

$p = \sec^2 x \qquad Q = \tan x.\sec^2 x$

$I.F = e^{\int p.dx} = e^{\int \sec^2 x.dx} = e^{\tan x}$

$\therefore$ solution of differential equation is

$y(I.F) = \int Q(I.F).dx + c$

$y.e^{\tan x} = \int \tan x.\sec^2 x.e^{\tan x}.dx + c$

$y.e^{\tan x} = I + C$---------(1)

when $I = \int e^{\tan x}.\tan x.\sec^2 x.dx$

put $\tan x = t \quad \Rightarrow \quad \sec^2 x.dx = dt$

$I = \int e^t.t.dt$

$I = t.e^t - \int e^t.dt$

$I = t.e^t - e^t$

$I = \tan x.e^{\tan x} - e^{\tan x}$--------(2)

substitute (2) in (1)

$y.e^{\tan x} = \tan x.e^{\tan x} - e^{\tan x} + c$

$y = \tan x - 1 + c.e^{-\tan x}$

Part E

VI. Answer the following question: 6 + 4 = 10

46. Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ hence evaluate $\int_0^{\pi/4} \log(1+\tan x)dx.$

Soln : Consider $\int_0^a f(a-x)dx$

When $x = a, \quad t = 0,$
and $x = 0, \quad t = a$

Put $a - x = t$ in RHS
$dx = -dt$

$= -\int_a^0 f(t)(dt) \quad = \int_0^a f(t)dt \qquad \left[\because \int_0^a f(x\,dx) = -\int_a^0 f(x)dx\right]$

$\int_0^a f(t)dx = \int_0^a f(x)dx$

$\therefore \quad \int_0^a f(x)dx = \int_0^a f(a-x)dx$

Let $I = \int_0^{\pi/4} \log(1+\tan x)dx.$

$$I=\int_0^{\pi/4}\log\left(1+\tan\left(\pi/4-x\right)\right)dx$$

$$I=\int_0^{\pi/4}\log\left(1+\frac{\tan\pi/4-\tan x}{1+\tan\pi/4\tan x}\right)dx$$

$$I=\int_0^{\pi/4}\log\left(\frac{1+\tan x+1-\tan x}{1+\tan x}\right)dx$$

$$I=\int_0^{\pi/4}\log\left(\frac{2}{1+\tan x}\right)dx$$

$$I=\int_0^{\frac{\pi}{4}}\log2-\int_0^{\frac{\pi}{4}}\log\left(1+\tan x\right)dx$$

$$I=\log2\int_0^{\pi/4}1\,dx\ -I$$

$$2I=\log2\left[x\right]_0^{\pi/4}$$

$$2I=\log2\left(\frac{\pi}{4}\right)$$

$$I=\frac{\pi}{8}\log2$$

OR **(6)**

2. Minimize and maximize $Z=5x+10y$

Subject to the constraints $x+2y\le120$

$x+y\ge60$

$x-2y\ge0$

$x\ge0\ ;\ y\ge0$

Soln: We have to minimize and maximize $Z=5x+10y$

Now, changing the given in equation $x+2y\le120$ ----------(1)

$x+y\ge60$ -------------(2)

$x-2y\ge0$ -------------(3)

$x\ge0\ ;\ y\ge0$-----------(4)

To equation.

$x+2y=120$

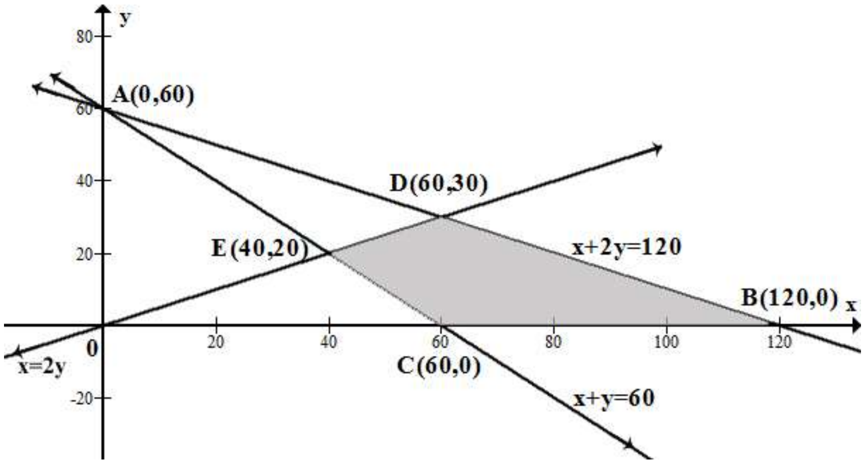
x	0	120
y	60	0

$x+y=60$

x	0	60
y	60	0

$x=2y$

x	0	20	40
y	0	10	20



The shaded region in the above figure is a feasible region determined by the system of constraints equation (1) to equation (4). It is observed that the feasible region is bounded. The co-ordinate of the corner point BDEC are, (120 , 0)., (60 , 30) (40 , 20) (60 , 0). The optimum value of Z are

Corner point	Z = 5x +10y
(120 , 0)	Z = 600 Maximum
(60 , 30)	Z = 600 Maximum
(40 , 20)	Z = 400
(60 , 0)	Z = 300 Minimum

$Z_{max} = 600$ at the points (60 , 30) , (120 , 0)
$Z_{min} = 300$ at the point (60 , 0)

Every point on line segment BD joining the two corner points B and D also gives same maximum value .

47. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$ and hence find A^{-1} **(4)**

Solution

$$A^2 = AA = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$$A^2 - 5A + 7I = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$A^2 - 5A + 7I = O$

Pre multiplied with A^{-1}

$A^{-1}.A^2 - 5A^{-1}A + 7A^{-1} = O$

$A^{-1}AA - 5\left(A^{-1}A\right) + 7A^{-1} = O$

$\left(A^{-1}A\right)A - 5\left(A^{-1}A\right) + 7\left(A^{-1}\right) = O$

$IA - 5I + 7A^{-1} = O$

$A - 5I + 7A^{-1} = 0$

$7A^{-1} = 5I - A$

$$A^{-1} = \frac{1}{7}(5I - A)$$

$$A^{-1} = \frac{1}{7}\left(5\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} - \begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}\right) = \frac{1}{7}\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$$

Thus, $A^{-1} = \frac{1}{7}\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$

OR

Find the value of k, if $\mathbf{f(x) = \begin{cases} \frac{k\cos x}{\pi - 2x} & \text{if } x \neq \pi/2 \\ 3 & \text{if } x = \pi/2 \end{cases}}$ **is continuous at** $\mathbf{x = \frac{\pi}{2}}$

Soln: The function is continuous at $x = \pi/2$.

$$\therefore \quad \lim_{x \to \frac{\pi}{2}} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \quad \lim_{x \to \frac{\pi}{2}} \frac{k\cos x}{\pi - 2x} = 3 \quad \Rightarrow \quad \lim_{x \to \frac{\pi}{2}} \frac{k\sin\left(\frac{\pi}{2} - x\right)}{2\left(\frac{\pi}{2} - x\right)} = 3$$

$$\text{as } x \to \frac{\pi}{2}, \quad \left(\frac{\pi}{2} - x\right) \to 0$$

$$\Rightarrow \quad \frac{k}{2} \lim_{\left(\frac{\pi}{2} - x\right) \to 0} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\left(\frac{\pi}{2} - x\right)} = 3$$

$$\Rightarrow \quad \frac{k}{2} \times 1 = 3$$

$$\therefore \quad K = 3 \times 2 = 6$$

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